Brief Introduction to Topological Strings

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This is a note on Gromov-Witten theory, following [MKKPTVVZ03, M05]. [1]

Contents

1 Topological sigma models

1.1 2d $\mathcal{N} = (2,2)$ sigma model

Let x^1, x^2 be Euclidean coordinates, and let $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$.

2d $\mathcal{N} = (2,2)$ SUSY algebra In 2d $\mathcal{N} = (2,2)$ supersymmetry, there are 4 supercharges, often denoted by $Q_{\alpha a}$ with $\alpha = \pm$ is the spinor (Lorentz) index and $a = \pm$ is the R-charge index.[2] The 2d $\mathcal{N} = (2,2)$ SUSY algebra satisfies the following relations (assuming no central charges) :

$$\{Q_{\alpha+}, Q_{\beta-}\} = \gamma^{\mu}_{\alpha\beta} P_{\mu}$$
$$\{Q_{\alpha\pm}, Q_{\beta\pm}\} = 0.$$

Commutation relations with some $U(1)$ currents are given by

$$[J, Q_{\pm a}] = \pm\frac{1}{2} Q_{\pm a}$$
$$[F_L, Q_{+\pm}] = \pm\frac{1}{2} Q_{+\pm}$$
$$[F_L, Q_{-\pm}] = 0$$
$$[F_R, Q_{+\pm}] = 0$$
$$[F_R, Q_{-\pm}] = \pm\frac{1}{2} Q_{-\pm}.$$

[1] This short note was prepared for string theory class April 22 and 24, 2019.

[2] We're following notation convention in [M05] here. In [MKKPTVVZ03], Q_α and $\overline{Q}_\alpha$ are used instead of $Q_{\alpha+}$ and $Q_{\alpha-}$, respectively.

Here J is the current for $SO(2)$ Lorentz transformation, and $F_{L,R}$ are left and right internal $U(1)$ (R-symmetry) currents.

Superspacetime formalism In 2d $\mathcal{N} = (2,2)$ superspacetime formalism, the spacetime is (locally) $\mathbb{R}^{2|(2,2)}$. Bosonic coordinates are as usual, but there are additional fermionic coordinates $\theta^{\alpha a}$, $\alpha = \pm$, $a = \pm$. Superfields are functions on this superspacetime. The covariant derivatives and the supercharges act on this superfields as follows :

$$D_{\alpha\pm} = \pm\frac{\partial}{\partial\theta^{\alpha\pm}} \mp \theta^{\alpha\mp}\partial_\alpha$$
$$Q_{\alpha\pm} = \pm\frac{\partial}{\partial\theta^{\alpha\pm}} \pm \theta^{\alpha\mp}\partial_\alpha.$$

They satisfy commutation relations

$$\{D_{\alpha+}, D_{\alpha-}\} = 2\partial_\alpha$$
$$\{Q_{\alpha+}, Q_{\alpha-}\} = -2\partial_\alpha$$
$$\{D_{\alpha a}, Q_{\beta b}\} = 0.$$

We can explicitly write down the supersymmetry transformation

$$\delta\Phi = \eta^{\alpha a}Q_{\alpha a}\Phi$$

in terms of components fields, but we won't do that here. (See [M05] p.73 instead.)

2d $\mathcal{N} = (2,2)$ sigma model Chiral multiplets are superfields Φ such that

$$D_{\alpha-}\Phi = 0.$$

Similarly anti-chiral multiplets $\overline{\Phi}$ satisfy

$$D_{\alpha+}\overline{\Phi} = 0.$$

Let's consider a collection of d chiral multiplets Φ^I and d anti-chiral multiplets $\Phi^{\bar{I}}$ with $I, \bar{I} = 1, \cdots, d$. In terms of component fields,

$$\Phi^I = x^I + \theta^{\alpha+}\psi^I_{\alpha+} + \theta^{-+}\theta^{++}F^I_{-+,++}$$
$$\Phi^{\bar{I}} = x^{\bar{I}} + \theta^{\alpha-}\psi^{\bar{I}}_{\alpha-} + \theta^{+-}\theta^{--}F^{\bar{I}}_{+-,--}$$

Consider an action with D-term only :

$$S = \int d^2z d^4\theta\, K(\Phi^I, \Phi^{\bar{I}}).$$

Assume that $\partial_I\partial_{\bar{J}}K$ is positive definite. Geometrically this is a sigma-model with complex d-dimensional Kähler target X. The local complex coordinate is given by $x^I, x^{\bar{I}}$. The fermions are spinors with values in

$$\psi_{\pm+} \in \Gamma(\Sigma_g, x^*TX^{(1,0)} \otimes S_\pm),$$
$$\psi_{\pm-} \in \Gamma(\Sigma_g, x^*TX^{(0,1)} \otimes S_\pm).$$

The Kähler potential is $K(x^I, x^{\bar{I}})$ and the Kähler metric is given by

$$G_{I\bar{J}} = \frac{\partial^2 K}{\partial x^I \partial x^{\bar{J}}}.$$

1.2 Topological twisting : A-twist and B-twist

When defined on a curved surface Σ_g, there is no covariantly constant spinor, and the supersymmetry is lost. However, By what is called *topological twisting*, we can preserve some supersymmetry in such a way that it agrees with the origina l theory on a flat surface.

Vector and axial R-symmetry The $F_{L,R}$ currents combine into F_V and F_A currents[3] :

$$\begin{aligned} F_V &:= F_L + F_R \\ F_A &:= F_L - F_R. \end{aligned}$$

It turns out that $U(1)_V$ is never anomalous, while $U(1)_A$ is anomalous. Recall that the kinetic fermion term of the action is

$$S_f = \int_{\Sigma_g} d^2z\, G_{\bar{I}J}(\psi^{\bar{I}}_{+-} D_{\bar{z}} \psi^J_{++} + \psi^{\bar{I}}_{--} D_z \psi^J_{-+}).$$

The axial anomaly is measured by the index of the Dirac operator[4] :

$$\dim \operatorname{Ker} D_{\bar{z}} - \dim \operatorname{Ker} D_z = \int_{\Sigma_g} x^*(c_1(TX)).$$

Hence the axial anomaly vanishes iff the target is Calabi-Yau.

Topological twisting (Topological) twisting is a procedure of redefinition of the spin of the fields using R-symmetries. There are two possible twists (up to conjugation) in our situation, called the *A-twist* and the *B-twist*. Those twists redefine the spin current as follows :

$$\begin{aligned} \text{A-twist :} \quad & \tilde{J} = J - F_V \\ \text{B-twist :} \quad & \tilde{J} = J + F_A. \end{aligned}$$

	$U(1)_E$	$U(1)_{F_L}$	$U(1)_{F_R}$	$U(1)_V$	$U(1)_A$	A-twist $U(1)'_E$	B-twist $U(1)'_E$
Q_{++}	$+1/2$	$+1/2$	0	$+1/2$	$+1/2$	0	$+1$
Q_{-+}	$-1/2$	0	$+1/2$	$+1/2$	$-1/2$	-1	-1
Q_{+-}	$+1/2$	$-1/2$	0	$-1/2$	$-1/2$	$+1$	0
Q_{--}	$-1/2$	0	$-1/2$	$-1/2$	$+1/2$	0	0

Table 1: Summary of quantum numbers of $Q_{\alpha a}$ under various $U(1)$ symmetries

In both cases we get two scalar supercharges and a vector supercharges. Define the *topological charge* to be

$$\begin{aligned} \text{A-twist :} \quad & Q_A = Q_{++} + Q_{--} \\ \text{B-twist :} \quad & Q_B = Q_{+-} + Q_{--}. \end{aligned}$$

Define a vector charge G_μ to be

$$\begin{aligned} \text{A-twist :} \quad & G_z = Q_{+-}\,,\, G_{\bar{z}} = Q_{-+} \\ \text{B-twist :} \quad & G_z = Q_{++}\,,\, G_{\bar{z}} = Q_{-+}. \end{aligned}$$

They satisfy the following ‘twisted SUSY’ relations :

$$\begin{aligned} Q^2 &= 0 \\ \{Q, G_\mu\} &= P_\mu. \end{aligned}$$

[3] We follow the convention of [M05], which is slightly different from [MKKPTVVZ03].
[4] See p.297 of [MKKPTVVZ03].

Cohomological field theory Let's briefly review the topological field theories of cohomological (a.k.a. Witten) type. A *cohomological field theory* is a quantum field theory on a manifold M that has a scalar symmetry δ acting on the fields in such a way that the correlation functions do not depend on the background metric. Two common features of cohomological field theories are :

- δ is a Grassmannian symmetry; i.e.
$$\delta^2 = 0.$$
- The energy momentum tensor is δ-exact; i.e.
$$T_{\mu\nu} = \delta G_{\mu\nu}$$
for some tensor $G_{\mu\nu}$.

The metric independence of correlation functions formally follows from the second condition, because for any δ-invariant operators $\mathcal{O}_1, \cdots, \mathcal{O}_n$,

$$\begin{aligned}\frac{\delta}{\delta g^{\mu\nu}}\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \langle \mathcal{O}_1 \cdots \mathcal{O}_n T_{\mu\nu} \rangle = \langle \mathcal{O}_1 \cdots \mathcal{O}_n \delta G_{\mu\nu} \rangle \\ &= \pm \langle \delta(\mathcal{O}_1 \cdots \mathcal{O}_n G_{\mu\nu}) \rangle.\end{aligned}$$

Here in the last line we have formally applied integration by parts and assumed that the boundary contribution vanishes. The observables in a cohomological field theory are δ-invariant operators, and the physical states are δ-invariant states. Observe that the twisted SUSY relations above already partially satisfy the conditions required for being a cohomological field theory. We will see that the twisted sigma models of type-A and B are indeed cohomological field theories.

1.3 Topological type-A model

Action The fermionic fields and the auxiliary fields change their spin after the twisting, so we rename them as follows :

$$\begin{aligned} \chi^I = \psi^I_{++}, \quad \rho^I_{\bar{z}} = \psi^I_{-+}, \quad F^I_{\bar{z}} = F^I_{-+,++}, \\ \chi^{\bar{I}} = \psi^I_{--}, \quad \rho^{\bar{I}}_{z} = \psi^{\bar{I}}_{+-}, \quad F^{\bar{I}}_{z} = F^{\bar{I}}_{+-,--}. \end{aligned}$$

The topological charge $Q = Q_A$ acts on scalar fields by [5]

$$[Q, x^i] = \chi^i, \quad \{Q, \chi^i\} = 0$$

The action for the theory is

$$\begin{aligned} S_A = \int_{\Sigma_g} d^2z\sqrt{g}\Big[G_{I\bar{J}}\Big(g^{\mu\nu}\partial_\mu x^I \partial_\nu x^{\bar{J}} - g^{\mu\nu}\rho^I_\mu D_\nu \chi^{\bar{J}} - g^{\mu\nu}\rho^{\bar{J}}_\mu D_\nu \chi^I - \frac{1}{2}g^{\mu\nu}\tilde{F}^I_\mu \tilde{F}^{\bar{J}}_\nu\Big) \\ + \frac{1}{2}g^{\mu\nu}R_{\bar{I}J\bar{K}L}\rho^{\bar{I}}_\mu \rho^J_\nu \chi^{\bar{K}}\chi^L\Big] = \{Q, V\} + \int_{\Sigma_g} x^*\omega \end{aligned}$$

where

$$V = \int_{\Sigma_g} d^2z\sqrt{g}g^{\mu\nu}G_{I\bar{J}}\left[\frac{1}{2}\rho^I_\mu \tilde{F}^{\bar{J}}_\nu + \frac{1}{2}\rho^{\bar{J}}_\mu \tilde{F}^I_\nu + (\rho^I_\mu \partial_\nu x^{\bar{J}} + \rho^{\bar{J}}_\mu \partial_\nu x^I)\right].$$

Hence the action is a sum of a Q-exact term and a topological term.[6] It follows that $T_{\mu\nu}$ is Q-exact. Therefore the twisted A-model is a topological field theory of cohomological type!

[5] For the full action, see p.79 of [M05] or p.409 of [MKKPTVVZ03].

[6] In [M05], it says that the A-model action is Q-exact, but it is wrong, as we see here that there's topological term.

Geometric interpretation Geometrically, we can interpret χ^i as the basis dx^i of differential forms on X. Then Q acts on x^i and χ^i like the de Rham differential on X. More generally,

$$\mathcal{O}_\phi = \phi_{i_1\cdots i_p}\chi^{i_1}\cdots\chi^{i_p} \leftrightarrow \phi = \phi_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$
$$Q_{++} \leftrightarrow \partial, \quad Q_{--} \leftrightarrow \bar\partial$$
$$Q_A = Q_{++} + Q_{--} \leftrightarrow d = \partial + \bar\partial.$$

It follows that

$$\{\text{physical operators}\} \simeq H^*_{dR}(X).$$

Semi-classical approximation For a cohomological field theory with Q-exact action, semi-classical approximation is exact. This is because

$$\frac{d}{dt}\langle\mathcal{O}\rangle(t) = \frac{d}{dt}\int \mathcal{D}\phi\mathcal{O}e^{-tS_A(\phi)} = \pm\langle\{Q,\mathcal{O}V\}\rangle = 0.$$

In the A-model, the action was not Q-exact but a sum of a Q-exact term and a topological term. Still we can first write the path integral as sum over topological sectors classified by

$$\beta = x_*[\Sigma_g] \in H_2(X,\mathbb{Z}).$$

Then for each topological sector we can apply the semi-classical approximation which localizes the path integral to instantons which are holomorphic maps $x : \Sigma_g \to X$.[7] In another words, the bosonic part of the action S_b is given by

$$\begin{aligned} S_b &= \int_{\Sigma_g} d^2z G_{I\bar J}(\partial_z x^I\partial_{\bar z}x^{\bar J} + \partial_z x^{\bar J}\partial_{\bar z}x^I) \\ &= 2\int_{\Sigma_g} d^2z G_{I\bar J}\partial_{\bar z}x^I\partial_z x^{\bar J} + \int_{\Sigma_g} x^*\omega \geq \int_{\Sigma_g} x^*\omega = \omega\cdot\beta \end{aligned}$$

where $\omega = iG_{I\bar J}dx^I\wedge dx^{\bar J}$ is the Kähler form. The minimum is attained for holomorphic maps. In case we have a non-trivial B-field, the action for a holomorphic map would be

$$S_b = \int_{\Sigma_g} x^*(\omega + iB) = (\omega + iB)\cdot\beta.$$

That is, we replace the Kähler form by the complexified Kähler form $\omega + iB$.

Correlation function Now let's consider the correlation function :

$$\langle\mathcal{O}_1\cdots\mathcal{O}_s\rangle = \sum_{\beta\in H_2(X,\mathbb{Z})}\langle\mathcal{O}_1\cdots\mathcal{O}_s\rangle_\beta$$

where

$$\langle\mathcal{O}_1\cdots\mathcal{O}_s\rangle_\beta = \int_{x_*[\Sigma_g]=\beta}\mathcal{D}x\mathcal{D}\chi\mathcal{D}\rho\, e^{-S}\mathcal{O}_1\cdots\mathcal{O}_s.$$

The semi-classical approximation localizes this integral to a finite dimensional space, namely the moduli space of holomorphic maps $\mathcal{M}_{\Sigma_g}(X,\beta)$. Its expected (complex) dimension is

$$\mathrm{vdim}\,\mathcal{M}_{\Sigma_g}(X,\beta) = \int_\beta c_1(T_X) + (\dim X)(1-g).$$

Let's identify the operator $\mathcal{O}_i$ inserted at $p_i \in \Sigma_g$ by the pull-back of $\phi_i \in H^*(X)$ by the evaluation map. Then correlation function is given by

$$\langle\mathcal{O}_1\cdots\mathcal{O}_s\rangle_\beta = e^{-t_\beta}\int_{\mathcal{M}_{\Sigma_g}(X,\beta)}\mathrm{ev}_1^*\phi_1\wedge\cdots\wedge\mathrm{ev}_s^*\phi_s$$

where $t_\beta = (\omega + iB)\cdot\beta$ is the complexified Kähler parameter. For example, when $g = 0$ and $\beta = 0$, $\mathcal{M}_{\Sigma_g}(X,\beta) \simeq X$, and we see that the correlation function is simply the classical intersection number.

[7] This directly follows from the localization principle saying that the path integral localizes to the loci where the Q-variation of the fermions vanishes. In the A-model, those Q-fixed points should obey $\partial_{\bar z}x = 0$ and hence are holomorphic.

Selection rule We have an obvious selection rule : the correlation function is non-vanishing only when the sum of the degrees of the operators matches with the dimension of the moduli space. This selection rule has a physical interpretation as well : an operator $\mathcal{O}_{\phi_i}$ corresponding to $\phi_i \in H^{p_i,q_i}(X)$ has vector R-charge $q_V = -p_i + q_i$ and axial R-charge $q_A = p_i + q_i$. The selection rule

$$\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = c_1(TX) \cdot \beta + \dim\ X(1-g)$$

follows from the fact that the vector R-symmetry is non-anomalous and the axial anomaly is measured by index of the Dolbeault operators, which is RHS of the selection rule.[8] Note that for Calabi-Yau X, the correlation function vanishes for $g > 1$. We'll see in the next section that we can still get meaningful invariants by coupling the theory to two-dimensional gravity.

Prepotential Suppose X is a Calabi-Yau 3-fold. The genus 0 Gromov-Witten invariants can be encoded into a generating function called the *prepotential* F_0 (a.k.a. the genus 0 partition function or genus 0 Gromov-Witten potential) of non-constant maps :

$$F_0(t) = \sum_{\beta \neq 0} N_{0,\beta} Q^\beta$$

where $Q_i = e^{-t_i}$ to emphasize its dependence on Kähler parameters. The coefficients $N_{0,\beta}$ are from the three-point functions

$$\langle \mathcal{O}_{\phi_1} \mathcal{O}_{\phi_2} \mathcal{O}_{\phi_3} \rangle = \int_X \phi_1 \cup \phi_2 \cup \phi_3 + \sum_{\beta \neq 0} Q^\beta N_{0,\beta} \int_\beta \phi_1 \int_\beta \phi_2 \int_\beta \phi_3$$

for $(1,1)$ forms ϕ_1, ϕ_2, ϕ_3. We can recover all the information about three-point (or higher) functions by differentiating the prepotential[9] :

$$\partial_i \partial_j \partial_k F_0(t) = \bar{\Gamma}_{ijk}(t) = \bar{C}_{ijk} - \int_X \phi_i \cup \phi_j \cup \phi_k.$$

Twisted chiral ring, an example : $X = \mathbb{CP}^1$ Let P and Q be operators corresponding to $1 \in H^0(\mathbb{CP}^1)$ and $H \in H^2(\mathbb{CP}^1)$. Then it is easy to see that[10]

$$\langle PPQ \rangle = 1$$
$$\langle QQQ \rangle = e^{-t}$$

where $t = (\omega + iB) \cdot [\mathbb{CP}^1]$. All the other correlation functions vanish. It follows that the twisted chiral ring of the $\mathbb{CP}^1$ sigma model is

$$QH^*(\mathbb{CP}^1) \simeq \frac{\mathbb{C}[x]}{(x^2 - e^{-t})}.$$

In general, the twisted chiral ring of the $\mathbb{CP}^n$ sigma model is

$$QH^*(\mathbb{CP}^n) \simeq \frac{\mathbb{C}[x]}{(x^{n+1} - e^{-t})}.$$

Observe that in the limit $t \to \infty$, the (small) quantum cohomology ring becomes the ordinary cohomology ring.

[8] In this sense, anomaly = dimension of moduli space.
[9] see p.533 of [MKKPTVVZ03].
[10] See pp.415–416 of [[MKKPTVVZ03]].

1.4 Topological type-B model

Action Let's assume that the target manifold X is Calabi-Yau. Let's rename the fields as their spin has changed after B-twist :

$$\rho_z^I = \psi_{++}^I, \quad \chi^{\bar{I}} = \psi_{+-}^{\bar{I}}, \quad F^I = F_{-+,++}^I,$$

$$\rho_{\bar{z}}^I = \psi_{-+}^I, \quad \bar{\chi}^{\bar{I}} = \psi_{--}^{\bar{I}}, \quad F^{\bar{I}} = F_{+-,--}^{\bar{I}}.$$

It is convenient to change the variables as

$$\eta^{\bar{I}} = \chi^{\bar{I}} + \bar{\chi}^{\bar{I}},$$

$$\theta_I = G_{I\bar{J}}(\chi^{\bar{J}} - \bar{\chi}^{\bar{J}}).$$

The $Q = Q_B$ acts on scalar fields by[11]

$$[Q, x^I] = 0, \quad [Q, x^{\bar{I}}] = \eta^{\bar{I}},$$

$$\{Q, \eta^{\bar{I}}\} = 0, \quad \{Q, \theta_I\} = G_{I\bar{J}}F^{\bar{J}}.$$

Note that Q_B acts differently on holomorphic and anti-holomorphic coordinates on X! The action for the theory is[12]

$$S_B = \int_{\Sigma_g} d^2z\, [\cdots] = \{Q, V\}$$

where V is given by

$$V = \int_{\Sigma_g} d^2z \sqrt{g}\, [\cdots] \,.$$

That is, the B-model action is Q-exact!

Geometric interpretation Geometrically, we can interpret $\eta^{\bar{I}}$ as the basis $dx^{\bar{I}}$ for the anti-holomorphic differential forms on X. Then Q acts on $x^I, x^{\bar{I}}, \eta^{\bar{I}}$ as the Dolbeault operator $\bar{\partial}$ on X. More generally,

$$\eta^{\bar{I}} \leftrightarrow dx^{\bar{I}}, \quad \theta_J \leftrightarrow \frac{\partial}{\partial x^J}$$

$$\mathcal{O}_\phi = \phi_{\bar{I}_1 \cdots \bar{I}_p}^{J_1 \cdots J_q} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_p} \theta_{J_1} \cdots \theta_{J_q} \leftrightarrow \phi_{\bar{I}_1 \cdots \bar{I}_p}^{J_1 \cdots J_q} dx^{\bar{I}_1} \wedge \cdots \wedge dx^{\bar{I}_p} \frac{\partial}{\partial x^{J_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{J_q}} \in \Omega^{0,p}(\wedge^q TX)$$

$$Q_B \leftrightarrow \bar{\partial}$$

It follows that the physical operators correspond to elements of the Dolbeault cohomology.

$$\{\text{physical operators}\} \simeq \bigoplus_{p,q=0}^{n} H^{0,p}(M, \wedge^q TM).$$

Correlation function The selection rule says that if ϕ_i is a (p_i, q_i)-form, then the correlation function $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle$ can be non-vanishing only when

$$\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = d(1-g).$$

[11] For the full action, see [M05] p.83 or [MKKPTVVZ03] p.420.

[12] The detail is not that important, so I won't write this down. See [M05] p. 84 instead.

Thanks to Q_B-exactness of the action, the semi-classical approximation is exact. In the B-model, there are no non-trivial instantons.[13] Hence it follows that the path integral reduces to an integral over X. The correlation function is given by

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \int_X \langle \phi_1 \wedge \cdots \wedge \phi_s, \Omega \rangle \wedge \Omega$$

where Ω is a non-vanishing section of $K_X = \Omega^{d,0}(X)$. This means that the correlation functions are not really functions but sections of a bundle on the moduli space of complex structures on the Calabi-Yau.

B-model prepotential The moduli space $\mathcal{M}$ of different complex structures on X has dimension $h^{2,1}$. Choose a symplectic basis (A_a, B^a), $a = 0, \cdots, h^{2,1}$ for $H_3(X)$. That is, $A_a \cap B^b = \delta_a^b$. Define the periods as

$$z_a = \int_{A_a} \Omega, \quad \mathcal{F}^a = \int_{B^a} \Omega.$$

Then it turns out that z^a are (locally) complex projective coordinates for the complex structure moduli $\mathcal{M}$. Of course we can introduce (local) inhomogeneous coordinates

$$t_a = \frac{z_a}{z_0}, \quad a = 1, \cdots, h^{2,1}.$$

The function

$$F_0(t_a) = \frac{1}{z_0^2} \frac{1}{2} \sum_{a=0}^{h^{2,1}} z_a \mathcal{F}^a$$

is called the B-model prepotential. In case X is a Calabi-Yau 3-fold, the three-point function of operators corresponding to Beltrami differentials μ_a, μ_b, μ_c (corresponding to tangent vectors $\frac{\partial}{\partial t_a}, \frac{\partial}{\partial t_b}, \frac{\partial}{\partial t_c}$) is

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \partial_a \partial_b \partial_c F_0.$$

Deformation of the theory Let me briefly comment on possible deformations on the theory. In cohomological field theory, we can deform the action by adding topological operators like

$$W_\phi^{(\gamma_n)} = \int_{\gamma_n} \phi^{(n)}$$

where $\gamma_n \in H_n(\Sigma)$ and $\phi^{(n)}$ is the n-th *topological descendant* of ϕ; i.e.

$$d\phi^{(n)} = \delta \phi^{(n+1)}.$$

In particular, in our 2-dimensional worldsheet, we can consider the second descendant. In order that this extra term we're adding to the action has vanishing $U(1)_V$ charge, ϕ should have degree 2. In case our target space X is Calabi-Yau 3-fold, this corresponds to a deformation of Kähler structure in A-model, and a deformation of complex structure in B-model.[14]

2 Topological string theory

2.1 Coupling to gravity

The correlation functions we have seen in the previous chapter are examples of Gromov-Witten invariants. For $g > 1$ the correlation functions were trivial, essentially because we were considering a fixed metric on the Riemann surface; the moduli space was too small. In order to get a non-trivial theory for higher genus we need to couple the theory to two-dimensional gravity.

[13]This again follows directly from the localization principle because Q-fixed points should obey $\partial_\mu x^I = 0$, meaning that it is a constant map.

[14]See [MKKPTVVZ03], pp.405-406 for more detail.

Topological string amplitude Define the genus g topological string amplitude (a.k.a. genus g free energy) for $g > 1$ as follows[15] :

$$F_g = \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} dm_i d\overline{m}_i \langle \prod_{i=1}^{3g-3} G_{++}(\mu_i) \prod_{i=1}^{3g-3} G_{--}(\overline{\mu}_{\overline{i}}) \rangle.$$

Here μ_i are $3g-3$ Beltrami differentials spanning the complex tangent space to $\mathcal{M}_g$ at the point Σ, dm_i are the dual one-forms to the μ_i, G_{++}, G_{--} are the currents corresponding to Q_{++} and Q_{--}; i.e.

$$T_{++}(z,\bar{z}) = \{Q, G_{++}(z,\bar{z})\}, \quad T_{--}(z,\bar{z}) = \{Q, G_{--}(z,\bar{z})\},$$

and

$$G_{++}(\mu_i) := \int_{\Sigma_g} d^2z\, G_{zz}\mu^z_{\bar{z}},$$
$$G_{--}(\overline{\mu}_{\overline{i}}) := \int_{\Sigma_g} d^2z\, G_{\bar{z}\bar{z}}\overline{\mu}^{\bar{z}}_z.$$

These G's have axial charge -1 and hence cancels axial anomaly. Therefore this provides an appropriate measure on the moduli space $\mathcal{M}_g$. This F_g depends only on the Kähler moduli for the type-A model, and on the complex moduli for the type-B model.

Relation to Calabi-Yau compactifications of type II string theory Recall that type II string theory compactified on a Calabi-Yau 3-fold X is a 4d $\mathcal{N} = 2$ theory. For type IIA string theory, the resulting theory has 1 gravity multiplet, $h^{1,1}(X)$ vector multiplets, and $1 + h^{2,1}(X)$ hypermultiplets. For type IIB string theory, the resulting theory has 1 gravity multiplet, $h^{2,1}(X)$ vector multiplets, and $1 + h^{1,1}(X)$ hypermultiplets. Notice that the number of vector multiplets agree with the dimension of the moduli that determine the prepotential $F_0(t)$ in the type-A and the type-B model. Indeed, from the target space point of view, $t_i(x)$ is be a 4-dimensional field which is a scalar component of a vector multiplet. It is known that topological string amplitudes on $\mathbb{R}^{3,1} \times X$ compute certain F-terms in the 4-dimensional effective action.

Type-A topological string The type-A topological string can be evaluated as a sum over instanton sectors, i.e. holomorphic curves. Hence we have

$$F_g(t) = \sum_\beta N_{g,\beta} Q^\beta.$$

Here $N_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{\mathrm{virt}}} 1 \in \mathbb{Q}$ are the *Gromov-Witten invariants*. This is exactly $I_{g,0,\beta}$ that we will discuss below.

2.2 Mathematical description of Gromov-Witten invariants

Moduli space $\overline{\mathcal{M}}_g$ of stable curves For $g > 1$, the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g is a non-singular Deligne-Mumford stack. By including stable nodal curves, we obtain its compactification $\overline{\mathcal{M}}_g$, the moduli space of stable curves.[16] This is a compact, connected, non-singular, irreducible Deligne-Mumford stack of (complex) dimension $3g-3$.[17]

Moduli space $\overline{\mathcal{M}}_{g,n}$ of stable pointed curves We can similarly compactify the moduli space $\mathcal{M}_{g,n}$ of n-pointed curves of genus g. Because each marked point gives one degree of freedom,

$$\dim \overline{\mathcal{M}}_{g,n} = \dim \overline{\mathcal{M}}_g + n = 3g - 3 + n.$$

[15]Due to axial charge violation, $F_g \in \Gamma(\mathcal{L}^{2g-2})$ where $\mathcal{L}$ is the line bundle on $\mathcal{M}_g$.

[16]A *stable curve* is a connected nodal curve such that every irreducible component of geometric genus 0 (rep. 1) has at least 3 (resp. 1) node branches. That is, it has finite group of automorphisms

[17]This dimension makes sense even for $g = 0, 1$ as virtual dimension.

Moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps Let X be a non-singular projective variety. A morphism f from a pointed nodal curve Σ to X is a stable map if every genus 0 contracted component of Σ has at least 3 special points. The moduli space of stable maps $(\Sigma, p_1, \cdots, p_n, f)$ (up to isomorphism) such that $f_*[\Sigma] = \beta \in H_2(X,\mathbb{Z})$ is denoted by $\overline{\mathcal{M}}_{g,n}(X,\beta)$. This is a compact Deligne-Mumford stack. Note that there are natural n evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ given by

$$\mathrm{ev}_i : (\Sigma, p_1, \cdots, p_n, f) \mapsto f(p_i).$$

The virtual dimension of the moduli space of stable maps is

$$\begin{aligned}\mathrm{vdim}\,\overline{\mathcal{M}}_{g,n}(X,\beta) &= h^0(\Sigma, f^*TX) - h^1(\Sigma, f^*TX) + \dim \mathrm{Def}(\Sigma, p_1, \cdots, p_n) - \dim \mathrm{Aut}(\Sigma, p_1, \cdots, p_n) \\ &= \int_\beta c_1(TX) + (\dim X)(1-g) + 3g - 3 + n = \int_\beta c_1(TX) + (\dim\, X - 3)(1-g) + n\end{aligned}$$

by the Riemann-Roch theorem. Notice that when X is Calabi-Yau 3-fold and $n = 0$, then this virtual dimension is always 0.

Virtual fundamental class It is known that the moduli space of stable maps carries a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \in H_{2\mathrm{vdim}}(X,\mathbb{Q})$.

Gromov-Witten invaraints We can pair the virtual fundamental class with cohomology classes to make numerical invariants of X. Given classes $\gamma_1, \cdots, \gamma_n \in H^*(X)$, the corresponding *Gromov-Witten invariant* is defined by

$$I_{g,n,\beta}(\gamma_1, \cdots, \gamma_n) = \langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n).$$

Similarly, the *(gravitational) descendant invariants* are defined by

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \psi_1^{a_1} \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) \cup \psi_n^{a_n}.$$

Here $\psi_i := c_1(\mathbb{L}_i)$ where $\mathbb{L}_i$ is the i-th tautological line bundle whose fiber at each point $(\Sigma, p_1, \cdots, p_n, f)$ is the cotangent line to Σ at p_i.

3 Further topics

3.1 Integrality : Donaldson-Thomas and Gopakumar-Vafa invariants

Gopakumar-Vafa invariants Gopakumar and Vafa made a remarkable conjecture that the topological string amplitude can be expressed in the following form :

$$F'(g_s, t) = \sum_{g=0}^{\infty} F'_g(t) g_s^{2g-2} = \sum_{\beta \neq 0} n^g_\beta g_s^{2g-2} \sum_{d \geq 1} \frac{1}{d} \left(\frac{\sin(dg_s/2)}{dg_s/2} \right)^{2g-2} Q^{d\beta}$$

where $n^g_\beta \in \mathbb{Z}$ are *Gopakumar-Vafa invariants* (BPS invariants). These BPS invariants count (with weight $(-1)^F$) the $SU(2)_L$ content of the number of BPS D2-branes with charge $\beta \in H_2(X,\mathbb{Z})$ in a particular basis of the $SU(2)_L$ representation ring. This conjecture in particular says that at genus 0 each BPS state contributes

$$\sum_{d=1}^{\infty} \frac{Q^{d\beta}}{d^3}.$$

This could be understood as a sum of contributions of all the multicoverings with degree d of a given primitive curve. The conjecture accounts to the bubbling effect as well. For instance, a genus 0 BPS state contributes to F_g with a weight

$$\frac{|B_{2g}|}{2g(2g-2)!}.$$

Apparently no rigorous definition of BPS invariants is known, but it is argued that

$$n_\beta^{g-\delta} = (-1)^{\dim(\mathcal{M}_{g,\delta,\beta})}\chi(\mathcal{M}_{g,\delta,\beta})$$

where $\mathcal{M}_{g,\delta,\beta}$ is the moduli space of irreducible genus g curves with δ ordinary nodes.

Donaldson-Thomas invariants Let X be a nonsingular projective Calabi-Yau 3-fold. Let $I_n(X,\beta)$ be the moduli space of ideal sheaves $\mathcal{I}$ satisfying

$$[Y] = \beta \in H_2(X,\mathbb{Z}), \quad \chi(\mathcal{O}_Y) = n$$

where Y is the subscheme of X determined by $\mathcal{I}$:

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

The Donaldson-Thomas invariant is the integration of the dimension 0 virtual fundamental class

$$\tilde{N}_{n,\beta} = \int_{[I_n(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Z}.$$

The partition function of the Donaldson-Thomas theory is

$$Z_{DT}(q,t) = \sum_{\beta,n} \tilde{N}_{n,\beta} q^n Q^\beta.$$

GW/DT correspondence Let X be a nonsingular projective Calabi-Yau 3-fold. Let $F'_{GW}(g_s,t) = \sum_{\beta\neq 0}\sum_{g\geq 0} N_{g,\beta} g_s^{2g-2} Q^\beta$ be the reduced free energy (contributions from non-constant maps). The reduced partition function is

$$Z'_{GW}(g_s,t) = \exp\ F'_{GW}(g_s,t) = 1 + \sum_{\beta\neq 0} Z'_{GW}(g_s)_\beta Q^\beta.$$

On the Donaldson-Thomas side, we have the reduced partition function

$$Z'_{DT}(q,t) = \frac{Z_{DT}(q,t)}{Z_{DT}(q)_0}$$

where $Z_{DT}(q)_0$ is the degree 0 partition function which is conjectured to be

$$\left(\prod_{n\geq 1} \frac{1}{(1-(-q)^n)^n}\right)^{\chi(X)}.$$

Maulik, Nekrasov, Okounkov and Pandharipande [MNOP03] conjectured that the two reduced partition functions are the same under the change of variables

$$Z'_{GW}(g_s,t) = Z'_{DT}(-e^{ig_s},t).$$

Gopakumar-Vafa conjecture implies that $Z'_{GW}(g_s)_\beta$ defines a series in $q = e^{-ig_s}$ with integer coefficients, and GW/DT correspondence identifies this q-series with the reduced partition function of the Donaldson-Thomas invariant.

3.2 Open Gromov-Witten invariants

One can extend the theory to the open topological strings. The worldsheet is now a Riemann surface $\Sigma_{g,h}$ with genus g and h holes. The relevant boundary conditions are Dirichlet boundary conditions on Lagrangian submanifolds $\mathcal{L} \subset X$. If we consider a topological open string theory with N topological D-branes wrapping a Lagrangian submanifold $\mathcal{L}$, then we also have $U(N)$ Chan-Paton degrees of freedom on the boundaries. The path integral is modified by inserting Wilson lines

$$\prod_i \operatorname{Tr} \operatorname{Pexp} \oint_{C_i} x^*(A)$$

corresponding to the boundaries. The type-A open topological string theory describes holomorphic maps from open Riemann surfaces $\Sigma_{g,h}$ to the Calabi-Yau with Dirichlet boundary conditions specified by $\mathcal{L}$. The topological sectors of an open string instanton can be classified by the bulk part and the boundary part. For the bulk part, we set

$$x_*[\Sigma_{g,h}] = \beta \in H_2(X, \mathbb{Z}).$$

For the boundary, we can specify the homology classes

$$f_*[C_i] = w_i \in H_1(\mathcal{L}).$$

In the end, the free-energy of the type-A open string theory at fixed genus g and boundary data w can be expressed as a sum over open string instantons :

$$F_{g,w}(t) = \sum_\beta F_{g,w,\beta} Q^\beta.$$

The quantities $F_{g,w,\beta}$ are *open Gromov-Witten invariants*.

Mathematical definition of open Gromov-Witten The basic difficulty in rigorously counting open holomorphic curves is that while degenerations of closed curves are of real codimension 2 in moduli, the boundary degenerations of curves with boundary are of real codimension 1 in moduli. Recent work of Ekholm and Shende [ES19] suggests that the open Gromov-Witten invariant should take value in the skein module of the Lagrangian brane. More precisely,

$$\Psi_{X,L} = 1 + \sum_{u\,\text{primitive}} (e^{\frac{1}{2}g_s} - e^{-\frac{1}{2}g_s})^{-\chi(u)} w(u) \cdot Q^{u_*[\Sigma]} \prod a_i^{\operatorname{lk}(u,L_i)} \langle \partial u \rangle \in \widehat{\operatorname{Sk}}(L)[[Q]].$$

Here $\operatorname{Sk}(L) := \bigotimes_{\mathbb{Q}_q} \operatorname{Sk}(L_i)$, $\mathbb{Q}_q = \mathbb{Z}[q^{\pm\frac{1}{2}}, (q^{\frac{n}{2}} - q^{-\frac{n}{2}})^{-1}]_{n=1}^\infty$, each $\operatorname{Sk}(L_i)$ is the $\mathbb{Q}_q[a^{\pm 1}]$-module generated by embedded framed 1-manifolds modulo isotopy and HOMFLY skein relations, and

$$\widehat{\operatorname{Sk}}(L) = \operatorname{Sk}(L) \otimes_{\mathbb{Q}_q} \mathbb{Q}((g_s))$$

with injective ring homomorphism determined by

$$q \mapsto e^{g_s}.$$

The factor $(e^{\frac{1}{2}g_s} - e^{-\frac{1}{2}g_s})^{-\chi(u)}$ accounts for degenerate contributions (multicovering). Ekholm and Shende showed how Ψ behaves well under conifold transition and also that the coefficient of $\ell_1 \otimes \cdots \otimes \ell_1$ in $\Psi_{T^*S^3, S^3 \cup L_K} = \Psi_{\tilde{X}, L_K}/\Psi_{\tilde{X}}$ is a monomial times $\langle K \rangle \in \operatorname{Sk}(S^3)$, i.e. the the HOMFLYPT polynomial of K.

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